Outer measure L^p spaces

25/05/2015.

(X, σ_1, S_1), and (X, σ_2, S_2). and (X, σ, S)

$\sigma \rightsquigarrow \mu \cdot (\mathbb{E})$., $\sigma_i \rightsquigarrow \mu_i$. $(\mathbb{E}_i)$.

Assume. $\mu \le \mu_1$, $\mu \le \mu_2$.

$$\forall E \in \mathbb{E},\ \exists E_1 \in \mathbb{E}_1,\ E_2 \in \mathbb{E}_2 \text{ s.t.}$$

$$S(f, E) \le S_1(f, E_1) \cdot S_2(f, E_2).$$

Outer Hölder estimate:

$$\|f_1 f_2\|_{L^p(X,\sigma,S)} \le 2 \cdot \|f_1\|_{L^{p_1}(X,\sigma_1,S_1)} \cdot \|f_2\|_{L^{p_2}(X,\sigma_2,S_2)}.$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Pf. W.l.o.g. assume $\|f_i\| = 1$, and fix $\lambda > 0$.

Pick. F_j: $\underset{F_j^c}{\operatorname{outsup}}\, S_j(f_j) \le \lambda^{p/p_j}$.

$$\mu_j(F_j) \le \mu_j(S_j(f_j) > \lambda^{p/p_j}) + \varepsilon.$$

$F = F_1 \cup F_2$.

W.t.s. $\mu(S(f_1 f_2) > \lambda) \le \mu(F)$.

$\forall E$, $\quad S(f_1 f_2 1_{F^c}, E) \le S_1(f_1 1_{F^c}, E_1) \cdot S_2(f_2 1_{F^c}, E_2)$.

$$\downarrow \qquad\qquad \downarrow$$

$$1_{F_2^c} \xleftrightarrow{\text{since}} 1_{F_2^c}.$$

$F_i \subset F$.

$$\le \lambda^{1/p_1} \lambda^{1/p_2} \le \lambda.$$

$\Rightarrow$ $\quad \mu(S(f_1 f_2) > \lambda) \le \mu(F) \le \mu_1(F_1) + \mu_2(F_2)$.

$$\le \sum_{j=1}^{2} \mu_j(S_1(f_j) > \lambda^{1/p_j}) + 2\varepsilon.$$

Back to paraproducts.

$T_i f(x,t) := \frac{1}{t}\varphi_i\left(\frac{\cdot}{t}\right) * f.$, $\quad \int \varphi_i = 0$. $\quad i = 1,2$.

$\Lambda(f_1, f_2, f_3) = \iint_{\mathbb{R}^2_+} (T_1 f_1)\cdot(T_2 f_2)\cdot(T_3 f_3) \frac{dx\,dt}{t}$. $\quad$ but $\int \varphi_3 \ne 0$ in general.

$$|\Lambda(f_1, f_2, f_3)| \lesssim \|(T_1 f_1)(T_2 f_2)(T_3 f_3)\|_{L^1(\mathbb{R}^2_+, \sigma, S_1)}.$$

{Recall $\mathbb{E} = \{\text{set of tents}\}$, $\sigma(T(x,t)) = t$ }

$S_1(f, T) := \frac{1}{t} \iint_T |f| \frac{dx\,dt}{t}$.

$\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{\infty} = 1\right)$

$$\cancel{\lesssim \prod_{j=1}^{3} \|T_j f_j\|_{L^{p_j}(\mathbb{R}^2_+, \sigma, S_j)}}$$

$$\lesssim \|T_1 f_1\|_{L^{p}(\mathbb{R}^2_+, \sigma, S_2)} \cdot \|T_2 f_2\|_{L^{p_2}(\mathbb{R}^2_+, \sigma, S_2)} \cdot \|T_3 f_3\|_{L^{p_3}(\mathbb{R}^2_+, \sigma, S_\infty)}.$$

Remark. Crucial matter is to find the right β_i's. The premeasure σ more or less remains fixed.

$$\lesssim \|f_1\|_{L^{p_1}(\mathbb{R})} \cdot \|f_2\|_{L^{p_2}(\mathbb{R})} \cdot \|f_3\|_{L^{p_3}(\mathbb{R})}.$$

Thm. $\|\mathcal{I}_F f\|_{L^p(X,\sigma,S_2)} \lesssim \|f\|_{L^p(\mathbb{R})}$. $X = \mathbb{R}^2_+$ or $\mathbb{R}^3_+$.

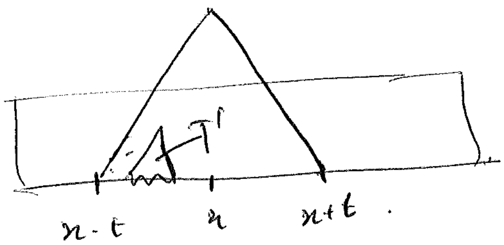
Tweaked for tents.

$\cancel{\text{If}}$ $\int \psi = 0$. $1 < p \le \infty$.

Pf. $p = \infty$: Calderón reproducing formula:

$$\iint_{\mathbb{R}^2_+} |\mathcal{I}_F f|^2 \frac{dx\,dt}{t} = c \cdot \int_{\mathbb{R}} |f|^2\,dx.$$

Need: $\mathcal{I}_F f \quad S_2(F,T) \lesssim \|f\|_\infty$ $\quad$ (T = tent).



eg: $g := f \cdot 1_{(x-3t,\,x+3t)}$.

$$\iint_{\mathbb{R}^2_+} \left|\tfrac{1}{t}\psi\left(\tfrac{\cdot}{t}\right) * g\right|^2 \frac{dx\,dt}{t} = c\int_{\mathbb{R}} |g|^2\,dx.$$

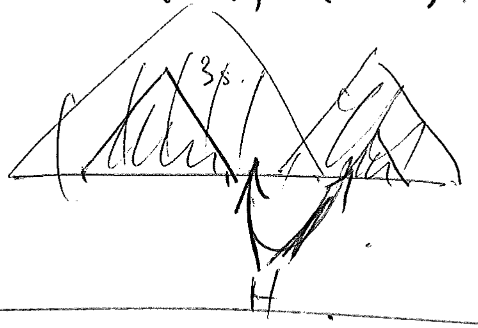
By ψ cptly spted, reduce to T'. Done.

Weak L^1: Need $\mu(S_2(F)>\lambda) \lesssim \frac{1}{\lambda}\|f\|_1$.

C.Z. decomp: $f = g + \sum b_i$, $\|g\|_\infty \le \lambda$,

supp $b_i \subset (x_i - s_i, x_i + s_i)$. $\int b_i = 0$.

$H := \bigcup_i T(x_i; 3s_i)$



$\boxed{\text{Need: mb } \sup_{H^c} S_2(\tilde{T} b) \lesssim \lambda. \quad \tilde{T} = \text{gradient}.}$ $(*)$

With this:

$$\mu(S_2(Tf)>\lambda) \le \underbrace{\mu(S_2(Tg)>\lambda)}_{\substack{= 0 \\ \text{since [illegible]} \le C\lambda}} + \underbrace{\mu(S_2(Tb)>\lambda)}_{\le \mu(H)}.$$

$\mu(H) \le \sum_i s_i \le \frac{1}{\lambda}\|f\|_1$. (by H-L $L_1 \to wL_1$).

~~$B_i(x) = \int_{-\infty}^{x} b_i$~~

~~Pf of $(*)$:~~

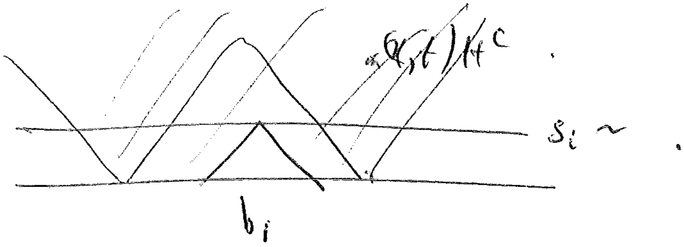
$B_i(x) = \int_{-\infty}^{x} b_i$.

$(\tilde{T} b_i)(x,t) = \frac{1}{t} B_i * \left(\frac{1}{t}\varphi'\left(\frac{\cdot}{t}\right)\right)$. φ' - derivative of φ.

~~$(x,t) \notin H \Rightarrow \tilde{T} b_i = 0$ if $t < s_i$~~

~~(see picture next~~

~~$\tilde{T}b(x,t)$~~



$(x,t) \notin H \Rightarrow \tilde{T}b_i = 0$ if $t < s_i$

$$|\tilde{T}b(x,t)| \lesssim \Big\| \frac{1}{t} \sum_{s_i<t} B_i \Big\|_\infty \underbrace{\|\varphi'\|_1}_{\lesssim 1}$$

$$\lesssim \underbrace{\frac{1}{t} \sup_i \|b_i\|}_{\lesssim 1 \text{ by C.Z.}}$$

$$\Rightarrow S_\infty(\tilde{T}b\, 1_{H^c}, \underset{\substack{\uparrow \\ \text{fixt}}}{E}) \lesssim 1$$

Estimate now S_i:

$$+ S_1(Tb_i 1_{H^c}, \underbrace{T(x,t)}_{=E})$$

$$= \iint_{T(x,t)\setminus H} |Tb_i| \frac{dy\,ds}{s}$$

$$\le \int_{s \ge s_i} \int_{|y-x_i| \le 2t} |Tb_i| \frac{dy\,ds}{s}$$

$$\lesssim \int_{s_i}^{\infty} \int_{|y-x_i| \le 2t} \|B_i\|_1 \|\varphi'\|_\infty \frac{dy\,ds}{s^3}$$

$$\lesssim \|B_i\|_1 \cdot t \frac{1}{s_i^2} \lesssim \|b_i\|_1 \cdot s_i \cdot \frac{t}{s_i^2}.$$

$$\Rightarrow \quad \delta_1(Tb\, 1_{H^c}, E) \lesssim \frac{1}{t} \lambda \sum_i \|b_i\|_1 \cdot s_i \cdot t$$

$$\lesssim \lambda \sum_i \|b_i\|_1 \cdot s_i .$$

Bilinear Hilbert Transform

Calderón problem: $|A(x)-A(y)| \leq c|x-y|$,

Cauchy Integral on graph of A.

$$f \mapsto \int_{\mathbb{R}} \frac{f(y)}{(x+iA(x)) - (y+iA(y))}\, dy \qquad L^2 \to L^2 \text{ bdd.}$$

1st attempt: Calderón 1st commutator.

$$f \mapsto p.v. \int \frac{A(x)-A(y)}{(x-y)^2} f(y)\, dy .$$

Problem solved by Coifman - Meyer - McIntosh 1982.

Failed attempt by Calderón:

$$\frac{A(x)-A(y)}{(x-y)^2} = \frac{1}{(x-y)} \int_0^1 A'(\alpha y + (1-\alpha)x)\, d\alpha . \qquad (y-x=t).$$

so, $$p.v.\left(\int \quad\right) = \int \frac{dt}{t} \int d\alpha \; A'(x+\alpha t) f(x+t).$$

$$= \int_0^1 d\alpha \left(p.v \int_{\mathbb{R}} f(x+t) A'(x+\alpha t) \frac{dt}{t} \right).$$

Q? $\left| \mathrm{p.v.} \int_{\mathbb{R}} f(x+t)\, A'(x+\alpha t) \frac{dt}{t} \right| \lesssim \|f\|_2 \|A'\|_\infty$?

$\alpha = 0$: ok.

$\alpha = 1$: ok.

$0 < \alpha < 1$: Lacey & Thiele '97.

~~Method of ...~~

Trilinear form $\Lambda(f_1, f_2, f_3) = \mathrm{p.v.} \int_{\mathbb{R}^2_x} \prod_j f_j(x - \beta_j t) \frac{dx\, dt}{t}$

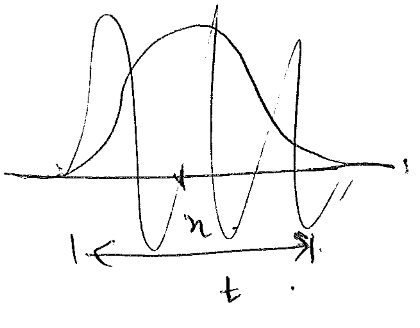
for $\beta = (1, -1, 0)$.

Th^{m} $|\Lambda(f_1, f_2, f_3)| \lesssim \|f_1\|_{L^{p_1}} \cdot \|f_2\|_{L^{p_2}} \cdot \|f_3\|_{L^{p_3}}$ if $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $p_i > 1$.

Three orth. vectors in $\mathbb{R}^3$.

$(1,1,1)$, $\beta = (1,-1,0)$, $\alpha = (1,1,-2)$.

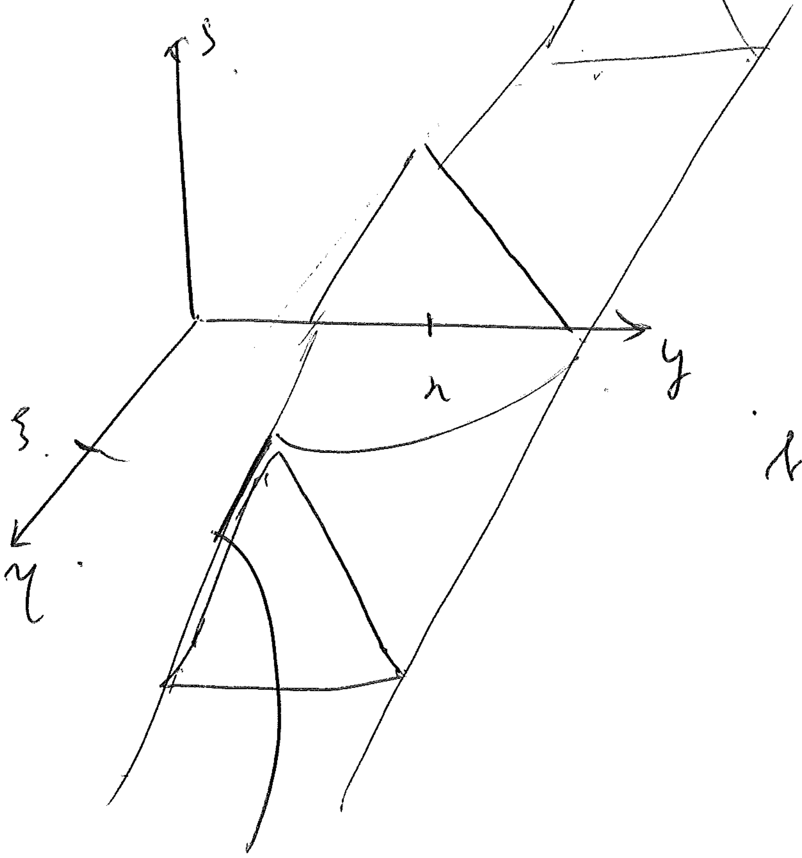
$$F(x, \xi, t) = \int_{\mathbb{R}} f(y) \cdot \frac{1}{t} \varphi\left(\frac{y-x}{t}\right) e^{i\xi(y-x)} dy.$$



wave packet.

$X = \mathbb{R}^3_+$.

Time-frequency tent $T(x, \xi, t) = \{ (y, \eta, s) : |y - x| < t - s,\ |\eta - \xi| \leq \frac{1}{s} \}$.



$\sigma(T_x(x, \xi, t)) = t.$

hizgé? $S_\infty + S_2$.